Deceptive Advertising with Rational Buyers

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ONLINE APPENDIX

In this Appendix we present in full additional results and extensions which are only mentioned in the paper. In the exposition we follow the order in which each topic first appears in the paper.

1 Pooling equilibria without advertising.

One may wonder whether there exist pooling equilibria in which neither seller advertises. Such equilibria would feature $t^\star = d^\star = 0$ and an equilibrium price $p^\star \leq \frac{\theta_l+\theta_h}{2}$ to satisfy the buyer's participation constraint. It is immediate to verify that such equilibria exist only if they are sustained by off-equilibrium beliefs of the type of **A1** — i.e., a seller charging a non-equilibrium price has to be deemed a low-quality one, regardless of any ad. Now, it turns out that these off-equilibrium beliefs are consistent with forward induction reasoning by the buyer only for equilibria entailing some advertising (like those we analyzed so far), while they violate the common refinement of Divinity 1 (**D1**) when the equilibrium entails no advertising at all. In particular, consider an equilibrium without advertising and suppose a buyer observes an ad that he should not observe in equilibrium. Then, by a simple forward induction argument, he should believe that the deviating seller provides high quality for sure. To see why, use the same logic of our "remark on equilibrium refinements" in the paper and suppose that, differently from **A1**, the buyer purchases with some probability, say α, from a deviating seller from whom he receives an ad, while he does not purchase from a deviating seller from whom he does not receive an ad, as in **A1**. We want to compare the equilibrium payoff of each seller at the pooling equilibrium without advertising to her off-equilibrium profits given the buyer's off-equilibrium behavior just described. Once again, denote by α^h (resp. α^l) the buyer's off-equilibrium strategy that makes a high-quality (resp. low-quality) seller indifferent between deviating and sticking to the equilibrium strategy. Moreover, denote by s the off-equilibrium advertising intensity and by $p^d \leq \theta_h$ the deviation price[1], both equal across sellers' types. Then it holds

$$\alpha^h p^d s - c(s) = \frac{p^\star}{2},$$
$$\alpha^l p^d s - c(s) - \phi s = \frac{p^\star}{2},$$

where equilibrium profits are on the right hand side. Clearly $\alpha^h < \alpha^l$. Hence, if the low-quality seller deviation profits are larger than equilibrium profits, this holds also for the high-quality seller but the opposite is not true. Thus, beliefs satisfying **D1** must assign probability one to the deviating seller having high quality, if the buyer receives an ad, and probability zero otherwise. The buyer will then purchase if he observes an ad he should not observe in equilibrium. This makes deviations profitable.

To see why the equilibrium with no advertising breaks down under off-equilibrium beliefs satisfying **D1**, suppose a high-quality seller deviates from the no advertising equilibrium by charging the equilibrium price $p^\star$, but advertising with intensity $s > 0$. Then, with probability s she reaches the buyer and she

[1] This may or may not be different from the equilibrium price.

earns $p^\star$, while she earns $\frac{p^\star}{2}$ with complementary probability $1-s$. The deviation has a sure cost $c(s)$ for the seller. Deviation profits will then be larger than equilibrium ones so long as

$$sp^\star + (1-s)\frac{p^\star}{2} - c(s) > \frac{p^\star}{2} \qquad \Longleftrightarrow \qquad s\frac{p^\star}{2} > c(s).$$

Because $c'(0) = 0$, by a straightforward application of de l'Hôpital there clearly exists an ϵ small but positive such that deviation profits are larger than equilibrium profits for $s < \epsilon$. Hence, we have shown that there has to be some advertising in equilibrium if the buyer's off-equilibrium beliefs satisfy **D1**.

Off-equilibrium beliefs like those in **A1** sustain pooling equilibria without advertising but, as just proved, are not consistent with forward induction, a minimal rationality criterion we have adopted throughout the paper. For this reason we deem such equilibria unappealing and we will not consider them in the analysis which follows.

2 Equilibrium Selection and Buyer's Welfare

When the model features both pooling and separating equilibria, destroying pooling equilibria with deceptive advertising by raising ϕ may not benefit the buyer. This is so, in particular, if sellers coordinate on the separating equilibrium (rather than on the pooling equilibrium without deception), and if the buyer's expected utility is lower at the separating than at the deceptive pooling equilibrium. In the online Appendix we show that there are cases where this can happen. Formally, this scenario emerges if

$$\pi^s > \max_{p^{**} \in (0, 2\phi]} \pi^{**}(p^{**}), \tag{1}$$

and

$$V^s < V^*(\overline{p}^*), \tag{2}$$

The next proposition shows that both conditions are satisfied in a non-empty region of parameters when the cost function is quadratic (see the Appendix to the paper for more on this cost function).

Proposition 1. *Suppose that $c(x) = kx^2/2$. A sufficient condition for both (1) and (2) to hold is*

$$\Delta > \max\left\{\frac{\phi}{2k}(4k-\phi), 4\frac{k^2}{k+\phi}\right\}.$$

In the separating equilibrium, the seller who offers the better product gets a premium that is increasing with the quality differential Δ — i.e., the high-quality seller exploits a sort of "monopolistic power" to extract more surplus from the buyer. When this premium is large enough, the buyer may actually prefer to be deceived with some probability rather than knowing the quality purchased in the separating equilibrium. Hence, when the game features three types of equilibria, a policy that completely prevents low-quality firms from airing false claims may actually do worse than a *laissez faire* approach.[2]

3 Endogenous sanctions

In this section we highlight an important policy implication of our model. We show that, if the (expected) sanction ϕ can be endogenously chosen by such an agency, the policy that maximizes the buyer's expected utility is more lenient than the policy that maximizes total (expected) welfare.

Suppose that a regulatory agency can enforce a sanction ϕ by paying the (increasing and convex) enforcement cost $e(\phi)$.[3] We make the following simplifying assumptions. First, we consider the 'regular' case in which the sellers' expected profits increase with the equilibrium price[4] while the buyer's expected utility decreases with it. Hence, sellers prefer deceptive to non-deceptive equilibria. Moreover, we posit that $e'(\cdot)$ is large enough that the Authority never finds it convenient to set ϕ so high to completely shut down equilibria with deceptive advertising.[5] Finally, we assume that $e''(\cdot)$ is large enough that the buyer's expected utility and total (expected) welfare, net of the enforcement cost, are single peaked with respect to ϕ.[6]

The Authority announces (and commits to) an expected sanction ϕ at the outset of the game. The timing is otherwise unchanged. Consider first a policy that maximizes the buyer's expected utility net of the enforcement cost — i.e.,

$$V^*(\overline{p}^*) - e(\phi) \equiv \mathrm{E}[\theta] - \overline{p}^* + \frac{\Delta}{2}\left[t^*(\overline{p}^*) - d^*(\overline{p}^*, \phi)\right] - e(\phi), \tag{3}$$

where, for the sake of clarity, we have made explicit the link between the advertising intensities, the equilibrium pooling price and the expected fine ϕ through the functions $t^*(\overline{p}^*)$ and $d^*(\overline{p}^*, \phi)$.

Differentiating with respect to ϕ, it is easy to verify that the expected sanction that maximizes (3) (say ϕ^b) solves the following first-order condition

$$\underbrace{\frac{\partial \overline{p}^*}{\partial \phi}\left[-1 + \frac{\Delta}{4}\frac{\partial}{\partial \overline{p}^*}\left[t^*(\overline{p}^*) - d^*(\overline{p}^*, \phi)\right]\right]}_{\geq 0} - \underbrace{\frac{\Delta}{4}\frac{\partial d^*(\overline{p}^*, \phi)}{\partial \phi}}_{<0} = e'(\phi), \tag{4}$$

Equation (4) shows that the impact of ϕ on the buyer's expected utility has two main effects (going

[2] Building on this intuition, in the next Section we show that, if the (expected) sanction ϕ is endogenously chosen by an agency concerned with consumer welfare, the policy that maximizes the buyer's expected utility is more lenient than the policy that maximizes total (expected) welfare. While interesting, this result goes beyond the scope of the present paper.

[3] Such cost reflects the Authority's prosecution effort or, in other words, the cost of building a given deterrence power.

[4] This means that sellers always prefer the pooling equilibrium with deceptive advertising to that with truthful advertising only, and that they charge the maximal pooling price. The result still holds if they coordinate on any price strictly lower than that.

[5] Formally, this requires $e'(\cdot)$ to be large enough at ϕ such that condition $\Delta \geq \underline{\Delta}(\phi, \theta_l)$ — i.e., the condition characterizing the main existence result stated in Proposition 1 of the paper — holds as equality. However, an alternative assumption is that ϕ cannot exceed the actual damage caused to the buyer, that is, the quality differential Δ. In fact, it is easy to verify that condition $\Delta \geq \underline{\Delta}(\phi, \theta_l)$ still holds at $\phi = \Delta$.

[6] It can be verified that all these conditions can be jointly satisfied in the quadratic example developed above.

in the same direction). First, increasing the cost of deceptive advertising tends to reduce the maximal pooling prices, which, other things being equal, makes the buyer better-off since we have assumed that his expected utility is decreasing in the equilibrium price. Second, when ϕ increases, the intensity of deceptive advertising chosen by the low-quality seller diminishes, which tends to increase the advertising premium and to make the buyer better-off. Clearly, the sum of these two effects needs to be traded off with the enforcement cost.

Next, consider an Authority that is concerned with total (expected) welfare — i.e., the sum of the buyer's expected utility and the sellers' expected profits plus the (expected) revenue ϕd^* collected from the low-quality sellers (which we assume to be redistributed to the society and not necessarily to the buyer through, e.g., the provision of public goods). Formally:

$$W\left(\overline{p}^*, \phi\right) = \mathrm{E}\left[\theta\right] + \frac{\Delta}{2}\left[t^*\left(\overline{p}^*\right) - d^*\left(\overline{p}^*, \phi\right)\right] - c\left(t^*\left(\overline{p}^*\right)\right) - c\left(d^*\left(\overline{p}^*, \phi\right)\right) - e\left(\phi\right).$$

Clearly, the equilibrium price has no direct impact on total welfare: it is just a monetary transfer between the buyer and the sellers. Differentiating with respect to ϕ, it is easy to verify that the sanction which maximizes total welfare (say ϕ^w) solves the following first-order condition

$$e'\left(\phi\right) = \frac{\partial \overline{p}^*}{\partial \phi}\left[\frac{\Delta}{4}\frac{\partial}{\partial \overline{p}^*}\left(t^*\left(\overline{p}^*\right) - d^*\left(\overline{p}^*, \phi\right)\right)\right] - \frac{\Delta}{4}\frac{\partial d^*\left(\overline{p}^*, \phi\right)}{\partial \phi} - \frac{\partial \overline{p}^*}{\partial \phi}\left[c'\left(t^*\left(\overline{p}^*\right)\right)\frac{\partial t^*\left(\overline{p}^*\right)}{\partial \overline{p}^*} + c'\left(d^*\left(\overline{p}^*, \phi\right)\right)\frac{\partial d^*\left(\overline{p}^*, \phi\right)}{\partial \overline{p}^*}\right] - c'\left(d^*\left(\overline{p}^*, \phi\right)\right)\frac{\partial d^*\left(\overline{p}^*, \phi\right)}{\partial \phi}.$$

A higher ϕ not only impacts the advertising premium (as discussed above), but it also affects the total costs of advertising. These costs decrease because the maximal pooling price $\overline{p}^*$ is (weakly) decreasing in ϕ, and the advertising intensities decrease if the equilibrium price becomes lower.

Proposition 2. *Buyer protection requires less enforcement than welfare maximization — i.e.,* $\phi^b < \phi^w$.

This result offers the surprising prediction that the more an Authority cares about buyers, the less it should protect them from deceptive advertising. The reason is that the buyer does not internalize the cost-saving effect of an increased sanction ϕ, which may induce sellers to invest less in advertising.

4 Extensions

This section provides two extensions of the baseline model. First, we show that the main qualitative features of the equilibrium with deceptive advertising characterized in the baseline model extend to a framework in which the sellers' types are imperfectly correlated. Second, we show that the game may also feature pooling equilibria with market breakdown — i.e., outcomes in which the buyer purchases one of the items on sale only in some states. We find sufficient conditions under which the main conclusions of the baseline model are with no loss of insights.

4.1 Weakly correlated types with private information

In this section we show that the characterization of the equilibria with deceptive advertising provided in Proposition 1 of the paper survives when the two types of sellers are not perfectly correlated.

We relax the assumption that types are perfectly negatively correlated by assuming that one seller provides a high-quality good, observable by the rival, and the rival has private information on its own good quality. The probability that the latter provides a low-quality good is ϱ. As before, the buyer cannot distinguish the quality of sellers. Notice that in this extended version of the game there are three types of sellers: the surely high-quality seller, denoted by H, the probabilistically high-quality seller, denoted by h, and the low-quality seller, denoted by l. Hence, with probability $(1-\varrho)$ both sellers provide high-quality goods and do not deceive the buyer. We denote their advertising strategies as t_H and t_h respectively. Finally, with probability ϱ there is a low-quality seller whose deceptive advertising strategy is denoted by d.

Even though this model is more general than the perfectly correlated type one, the results are substantially the same. We summarize them in the following proposition.

Proposition 3. *When the sellers' types are not perfectly (negatively) correlated there exists a set of (symmetric) weak PBE in which trade occurs with certainty, sellers charge the same price (p^*), and the low-quality firm deceives the customer ($d^* > 0$) if and only if $\Delta > \underline{\Delta}(\phi, \theta_l)$. The equilibrium characterization is identical to that of Proposition 1 of the paper, which also reports the expression for $\underline{\Delta}(\phi, \theta_l)$. Both sellers charge a price $p^* \in [2\phi, \overline{p}^*]$. The maximal pooling price that can be charged in these equilibria is:*

$$\overline{p}^* = \min \{p(\phi, \Delta), \mathrm{E}[\theta]\}$$

where $p(\phi, \Delta)$ is defined exactly as in Proposition 1 of the paper.

For any $p^* \in [2\phi, \overline{p}^*]$, the equilibrium advertising strategies are such that $t_H^* = t_h^* = t^* > d^* \geq 0$ and t^* and d^* satisfy the first-order conditions stated in Lemma 3 of the paper. The buyer's equilibrium strategy is the same as that stated in Lemma 2 of the paper.

Hence, the characterization of deceptive equilibria is robust to the introduction of imperfect correlation between sellers' types. The only notable difference is that the equilibrium profit of the surely high-quality seller, type H, is now increasing in the probability, ϱ, that the opponent sells a low-quality item — i.e. has type l.[7] This is because a low-quality seller advertises less strongly in equilibrium than a high-quality seller, thereby increasing the chances that the surely high-quality seller, H, makes the sale. Indeed, as ϱ tends to 1, the model reverts to the perfectly correlated one and equilibrium profits converge to those characterized in the paper, that is

$$\pi_h^*(p^*) = p^* \frac{1 + t^* - d^*}{2} - c(t^*),$$

[7] See the Appendix for details of this comparative statics.

and

$$\pi_l^*(p^*) = p^* \frac{1 + d^* - t^*}{2} - c(d^*) - \phi d^*.$$

4.2 Equilibria with market breakdown

In this section we briefly discuss the class of equilibria in which the buyer refrains from purchasing in some states. We say that in these cases the market breaks down. For brevity, we consider again the quadratic example and assume that sellers' qualities are perfectly (negatively) correlated. Out-of-equilibrium beliefs are as in **A1**. We want to show that there exists a non-empty region of parameters in which equilibria with market breakdown do not exist, but the game features pooling equilibria with deceptive advertising.

Consider a candidate equilibrium in which the low-quality seller's deceptive advertising is $\tilde{d} > 0$ while the high-quality seller invests $\tilde{t}$ in truthful advertising. Let $\tilde{p}$ be the equilibrium price. Suppose that in this equilibrium the market breaks down in some states of nature — i.e., there exists at least a pair of signals (s_i, s_j) such that

$$\tilde{p} > \mathrm{E}[\theta | s_i, s_j],$$

otherwise trade would occur with probability 1. Given the buyer's posteriors defined in equations (1)-(4) in the paper, this outcome can occur only if

$$\tilde{p} > \mathrm{E}[\theta | \varnothing, \varnothing] = \mathrm{E}[\theta | h, h] = \theta_l + \frac{\Delta}{2},$$

and $\tilde{p} \leq \mathrm{E}[\theta | h, \varnothing] < \theta_h$, otherwise there would be full market breakdown (which clearly cannot be an equilibrium outcome).

Following the logic developed throughout the paper, it is easy to show that in every (symmetric) pooling equilibrium with market breakdown the buyer's strategy satisfies the following properties: (*i*) symmetry — i.e., $\tilde{\alpha}_i(s, s') = \tilde{\alpha}_j(s, s')$ for every (s, s'); (*ii*) when the buyer receives only one ad he buys from the seller who has aired that ad — i.e., $\tilde{\alpha}_i(h, \varnothing) = 1$ (resp. 0) if and only if $\tilde{t} > \tilde{d}$ (resp. $<$); and (*iii*) when the buyer receives two identical signals he does not purchase the item — i.e., $\tilde{\alpha}_i(s, s) = 0$.

The economic intuition of this result is as in Lemma 2 of the paper: since sellers are *ex-ante* identical, a symmetric equilibrium exists if and only if the buyer treats them symmetrically at the interim stage. The only remarkable difference is point (*iii*): the buyer does not purchase the item when he receives two identical signals. Hence, for any price $\tilde{p}$ charged in equilibrium, the sellers' profits are

$$\begin{aligned} \tilde{\pi}_h(\tilde{p}) &= \tilde{p}(1 - \tilde{d})\tilde{t} - \frac{k}{2}\tilde{t}^2, \\ \tilde{\pi}_l(\tilde{p}) &= \tilde{p}(1 - \tilde{t})\tilde{d} - \frac{k}{2}\tilde{d}^2 - \phi \tilde{d}. \end{aligned}$$

The first-order conditions with respect to $\tilde{t}$ and $\tilde{d}$ are, respectively

$$\begin{aligned} \tilde{p}(1-\tilde{d}) &= k\tilde{t}, \\ \tilde{p}\left(1-\tilde{t}\right) &= k\tilde{d}+\phi. \end{aligned}$$

These conditions highlight an important difference with the analysis developed so far: truthful and deceptive advertising are strategic substitutes — i.e., when $\tilde{d}$ increases, $\tilde{t}$ must decrease and *vice versa*.

To rule out equilibria with corner solutions we impose the intuitive sufficient condition $\theta_h < k$, so that $\tilde{p} < \theta_h < k$. This implies that at an interior solution

$$\begin{aligned} \tilde{t} &= \frac{\tilde{p}}{k+\tilde{p}} + \frac{\tilde{p}\phi}{(k-\tilde{p})(k+\tilde{p})}, \\ \tilde{d} &= \frac{\tilde{p}}{k+\tilde{p}} - \frac{\phi k}{(k-\tilde{p})(k+\tilde{p})}, \end{aligned}$$

where it can be readily verified that $1 > \tilde{t} > \tilde{d}$. An interesting difference with the previous analysis is that $\tilde{d} > 0$ if and only if $\tilde{p} \in (\tilde{p}_0, \tilde{p}_1)$, with

$$\tilde{p}_0 = \frac{k-\sqrt{k\left(k-4\phi\right)}}{2} < \frac{k+\sqrt{k\left(k-4\phi\right)}}{2} = \tilde{p}_1,$$

which are defined only if $\phi < k/4$. Hence, the low-quality seller has an incentive to invest in deceptive advertising if and only if the equilibrium price is neither too low nor too high. The reason is simple. If the equilibrium price is too small, the low-quality seller has no incentive to invest in deceptive advertising (as seen before). However, when the market breaks down, the sellers' advertising choices are strategic substitutes. Hence, if the equilibrium price is too high, the high-quality seller's investment in truthful advertising crowds out the low-quality seller's incentive to advertise. Therefore, in order to have deceptive advertising in equilibrium, the pooling price must be not too large. Equilibrium profits are

$$\begin{aligned} \tilde{\pi}_h\left(\tilde{p}\right) &= \frac{k\tilde{p}^2}{2\left(k+\tilde{p}\right)^2\left(\tilde{p}-k\right)^2}\left(k-\tilde{p}+\phi\right)^2, \\ \tilde{\pi}_l\left(\tilde{p}\right) &= \frac{k\tilde{p}^2}{2\left(k+\tilde{p}\right)^2\left(\tilde{p}-k\right)^2}\left[k-\tilde{p}-\frac{k}{\tilde{p}}\phi\right]^2, \end{aligned}$$

where it is easy to verify that $\tilde{\pi}_h\left(\tilde{p}\right) > \tilde{\pi}_l\left(\tilde{p}\right)$ in the region of parameters under consideration. Hence, the equilibrium characterization follows the same steps as the characterization of pooling equilibria with deceptive advertising — see Section 5 in the paper. That is, the relevant incentive constraint to consider is that of the low-quality seller — i.e.,

$$\tilde{\pi}_l\left(\tilde{p}\right) = \frac{k\tilde{p}^2}{2\left(k+\tilde{p}\right)^2\left(\tilde{p}-k\right)^2}\left[k-\tilde{p}-\frac{k}{p}\phi\right]^2 \geq \pi^d(\tilde{p}) = \tilde{p}-\Delta. \tag{5}$$

As long as this inequality defines a non-empty set, the outcome described so far is a *weak* PBE.

Notice that $\tilde{\pi}_l(\tilde{p}) = 0$ for $\tilde{p} = \tilde{p}_0$. Moreover, in the region of parameters under consideration, it can be verified that $\tilde{\pi}_l(\tilde{p})$ is an increasing function of $\tilde{p}$ and $\partial\tilde{\pi}_l(\tilde{p})/\partial\tilde{p} < 1$. Hence, a sufficient condition under which equation (5) defines an empty set is,

$$\Delta < \tilde{\Delta}(\phi) \equiv \frac{k - \sqrt{k(k-4\phi)}}{2},$$

meaning that $\pi^d(\tilde{p}) > 0$ at $\tilde{p} = \tilde{p}_0$. This suggests that, as long as $\underline{\Delta}(\phi, \theta_l) < \tilde{\Delta}(\phi)$, there exists a non empty region of parameters such that the game features a pooling equilibrium in which trades occurs with certainty, but not an equilibrium with market breakdown. For example, if

$$\underline{\Delta}(\phi, \theta_l) = 2(2\phi - \theta_l),$$

it is easy to verify that $\tilde{\Delta}(\phi) - \underline{\Delta}(\phi, \theta_l) = 2\theta_l > 0$. Hence, in this region of parameters the analysis developed so far apply without loss of insights.

Proofs

Proof of Proposition 1. Let's find first a sufficient condition for (1). Notice that

$$\pi^{**}(p^{**}) = \frac{p^*}{2}\left(1 - \frac{p^*}{8k}\right)$$

and

$$\max_{p^{**}\in(0,2\phi]} \pi^{**}(p^{**}) = \phi\left(1 - \frac{\phi}{4k}\right).$$

So, recalling that $\pi^s = \frac{\Delta}{2}$,

$$\pi^s > \max_{p^{**}\in(0,2\phi]} \pi^{**}(p^{**}) \quad \Leftrightarrow \quad \Delta > \frac{\phi}{2k}(4k-\phi),$$

which defines a lower bound for Δ.

Let's now turn to the sufficient condition for (2). Notice that $V^s = \theta_l$, while

$$\begin{aligned} V(\overline{p}^*) &= \mathrm{E}[\theta] - \overline{p}^* + \frac{\Delta}{2}\frac{\phi}{k} \\ &= \theta_l + \frac{\Delta}{2}\left(1+\frac{\phi}{k}\right) - \overline{p}^* \\ &> \theta_l + \frac{\Delta}{2}\left(1+\frac{\phi}{k}\right) - 2k \end{aligned}$$

where the inequality comes from the Inada conditions (in fact, as shown in the 'Quadratic cost function' paragraph at the end of the Appendix, $t^* = \frac{p^*}{2k} < 1 \Leftrightarrow p^* < 2k$). Thus, a sufficient condition for $V^s < V(\overline{p}^*)$ is

$$\theta_l < \theta_l + \frac{\Delta}{2}\left(1+\frac{\phi}{k}\right) - 2k \quad \Leftrightarrow \quad \Delta > 4\frac{k^2}{k+\phi},$$

which defines another lower bound for Δ. Thus, conditions (1) and (2) are jointly met if

$$\Delta > \max\left\{\frac{\phi}{2k}(4k-\phi), \frac{4k^2}{k+\phi}\right\}.$$

this concludes the proof. ■

Proof of Proposition 2. To begin with, notice that from the first-order condition (4)

$$e'\left(\phi^b\right) = \frac{\partial \overline{p}^*}{\partial \phi}\left[-1 + \frac{\Delta}{4}\frac{\partial}{\partial \overline{p}^*}\left[t^*(\overline{p}^*) - d^*(\overline{p}^*, \phi)\right]\right] - \frac{\Delta}{4}\frac{\partial d^*(\overline{p}^*, \phi)}{\partial \phi}\bigg|_{\phi=\phi^b}.$$

Next, substituting this expression into the first-order condition of the total welfare maximization problem evaluated at ϕ^b one can immediately verify that

$$\left.\frac{dW(\overline{p}^*,\phi)}{d\phi}\right|_{\phi=\phi^b} = \left.-2\frac{\partial\overline{p}^*}{\partial\phi}\frac{\partial\pi^*(\overline{p}^*,\phi)}{\partial p^*} - c'(d^*(\overline{p}^*,\phi))\frac{\partial d^*(\overline{p}^*,\phi)}{\partial\phi}\right|_{\phi=\phi^b}.$$

Recall that we have assumed that $\partial\pi^*(p^*,\phi)/\partial p^* > 0$ for every p^*. Moreover, from the first-order condition (7) in the paper it follows that $\partial d^*(\overline{p}^*,\phi)/\partial\phi < 0$. Hence, since $\partial\overline{p}^*/\partial\phi \leq 0$ from Corollary 1 of the paper, it follows that

$$\left.\frac{dW(\overline{p}^*,\phi)}{d\phi}\right|_{\phi=\phi^b} > 0.$$

Hence, by concavity of $W(\overline{p}^*,\phi)$ with respect to ϕ, it follows that $\phi^b < \phi^w$. ■

Proof of Proposition 3. Given that we wish to prove existence, we simplify the proof of this extension with respect to that of Proposition 1 in the paper. We assume that the equilibrium is that described in the proposition and then prove that actually it is an equilibrium. By Bayes' rule and using symmetry, the equilibrium beliefs are

$$\begin{aligned}
\Pr(\theta_i=\theta_l|h,h,\mathbf{p}^*) &= \frac{1}{2}\frac{\varrho d^*}{(1-\varrho)t_h^*+\varrho d^*},\\
\Pr(\theta_i=\theta_l|\varnothing,h,\mathbf{p}^*) &= \frac{\varrho t_H^*(1-d^*)}{\left(t_h^*(1-\varrho)+\varrho d^*\right)\left(1-t_H^*\right)+\left(1-t_h^*(1-\varrho)-\varrho d^*\right)t_H^*},\\
\Pr(\theta_i=\theta_l|h,\varnothing,\mathbf{p}^*) &= \frac{\varrho(1-t_H^*)d^*}{\left(t_h^*(1-\varrho)+\varrho d^*\right)\left(1-t_H^*\right)+\left(1-t_h^*(1-\varrho)-\varrho d^*\right)t_H^*},\\
\Pr(\theta_i=\theta_l|\varnothing,\varnothing,\mathbf{p}^*) &= \frac{1}{2}\frac{\varrho(1-d^*)}{1-(1-\varrho)t_h^*-\varrho d^*}.
\end{aligned}$$

Notice that

$$\Pr(\theta_i=\theta_l|h,\varnothing,\mathbf{p}^*) < \Pr(\theta_i=\theta_l|\varnothing,h,\mathbf{p}^*), \tag{P1}$$

if and only if $t_H^* > d^*$. That is, it is more likely that the seller has low quality if she does not post an ad, rather than if she does, provided that the surely high-quality seller does more advertising than the low-quality one. If this last condition holds, it is more likely that a seller is of bad quality if the buyer does not observe any ad rather than in the case he observes an ad from each seller, that is,

$$\Pr(\theta_i=\theta_l|\varnothing,\varnothing,\mathbf{p}^*) > \Pr(\theta_i=\theta_l|h,h,\mathbf{p}^*).$$

Let us now determine the optimal advertising strategies of the sellers. Given the behavior strategy of the buyer, expected profits of a seller with type, respectively, H, h and l are

$$\begin{aligned}
\pi_H &= \frac{1}{2}(1+t_H-\varrho d-(1-\varrho)t_h)p - c(t_H),\\
\pi_h &= \frac{1}{2}(1+t_h-t_H)p - c(t_h),\\
\pi_l &= \frac{1}{2}(1+d-t_H)p - c(d) - \phi d,
\end{aligned}$$

and the system of first-order conditions is

$$c'\left(t_H^*\right) = \frac{p^*}{2}, \qquad c'\left(t_h^*\right) = \frac{p^*}{2}, \qquad c'\left(d^*\right) = \frac{p^*}{2} - \phi,$$

it turns out that it must be

$$c'\left(t^*\right) = \frac{p^*}{2}, \qquad c'\left(d^*\right) = \frac{p^*}{2} - \phi,$$

where $t_H^* = t_h^* = t^*$. Notice that $t^* > d^*$, and hence (P1) is satisfied. Thus, equilibrium profits are

$$\begin{aligned}
\pi_H^* &= \frac{p^*}{2}\left(1 + \varrho\left(t^* - d^*\right)\right) - c\left(t^*\right), \\
\pi_h^* &= \frac{p^*}{2} - c\left(t^*\right), \\
\pi_l^* &= \frac{p^*}{2}\left(1 - t^* + d^*\right) - c\left(d^*\right) - \phi d^*,
\end{aligned}$$

with $\pi_H^* > \pi_h^* > \pi_l^*$. The first inequality is trivial. As for the second, notice that using repeatedly the first-order conditions for the advertising strategies we have

$$\pi_h^* - \pi_l^* = -c\left(t^*\right) + c'\left(t^*\right) t^* + c\left(d^*\right) - c'\left(d^*\right) d^* > 0,$$

where the last inequality is equivalent to

$$c\left(d^*\right) - c'\left(d^*\right) d^* > c\left(t^*\right) - c'\left(t^*\right) t^*,$$

which is satisfied because $t^* > d^*$ and $c\left(x\right) - xc'\left(x\right)$ is decreasing in x — since its first derivative is: $-c''\left(x\right) x < 0$.

Now we have to check the buyer's strategies. Given the assumption of symmetry, whenever the buyer receives the same signal (high-quality or no signal) he is indifferent between the two sellers. Hence, any probability distribution about purchases is an equilibrium one, provided that the participation constraint is satisfied. In particular, $\alpha_i^*\left(h, h\right) = \alpha_i^*\left(\varnothing, \varnothing\right) = \frac{1}{2}$ for every $i = 1, 2$ is an equilibrium strategy. If the buyer receives an ad from only one seller the buyer will purchase from that seller, since (P1) holds.

Given the above results (i.e., $\pi_H^* > \pi_h^* > \pi_l^*$) and deviation profits which are also in this case $\pi^d = p^* - \Delta$ for any type of seller, the only participation and incentive compatibility constraints to be checked are those of l. Thus, noticing that the equilibrium profit of the l type is unchanged relative to the perfect correlation case, the characterization is identical to that of Proposition 1 of the paper. Notice however that equilibrium beliefs are now different and more importantly profits of type H are increasing in ϱ (since d^* is independent from ϱ). ■